Module 1: Elasticity

1.1.1 INTRODUCTION

If the external forces producing deformation do not exceed a certain limit, the deformation disappears with the removal of the forces. Thus the elastic behavior implies the absence of any permanent deformation. Elasticity has been developed following the great achievement of Newton in stating the laws of motion, although it has earlier roots. The need to understand and control the fracture of solids seems to have been a first motivation. Leonardo da Vinci sketched in his notebooks a possible test of the tensile strength of a wire. Galileo had investigated the breaking loads of rods under tension and concluded that the load was independent of length and proportional to the cross section area, this being the first step toward a concept of stress.

Every engineering material possesses a certain extent of elasticity. The common materials of construction would remain elastic only for very small strains before exhibiting either plastic straining or brittle failure. However, natural polymeric materials show elasticity over a wider range (usually with time or rate effects thus they would more accurately be characterized as viscoelastic), and the widespread use of natural rubber and similar materials motivated the development of finite elasticity. While many roots of the subject were laid in the classical theory, especially in the work of Green, Gabrio Piola, and Kirchhoff in the mid-1800's, the development of a viable theory with forms of stress-strain relations for specific rubbery elastic materials, as well as an understanding of the physical effects of the nonlinearity in simple problems such as torsion and bending, was mainly the achievement of the British-born engineer and applied mathematician Ronald S. Rivlin in the1940's and 1950's.

1.1.2 THE GENERAL THEORY OF ELASTICITY

Linear elasticity as a general three-dimensional theory has been developed in the early 1820's based on Cauchy's work. Simultaneously, Navier had developed an elasticity theory based on a simple particle model, in which particles interacted with their neighbours by a central force of attraction between neighboring particles. Later it was gradually realized, following work by Navier, Cauchy, and Poisson in the 1820's and 1830's, that the particle model is too simple. Most of the subsequent developments of this subject were in terms of the continuum theory. George Green highlighted the maximum possible number of independent elastic moduli in the most general anisotropic solid in 1837. Green pointed out that the existence of elastic strain energy required that of the 36 elastic constants relating the 6 stress components to the 6 strains, at most 21 could be independent. In 1855, Lord Kelvin showed that a strain energy function must exist for reversible isothermal or adiabatic response and showed that temperature changes are associated with adiabatic elastic deformation. The middle and late 1800's were a period in which many basic elastic solutions were derived and applied to technology and to the explanation of natural phenomena. Adhémar-Jean-Claude Barré de Saint-Venant derived in the 1850's solutions for the torsion of noncircular cylinders, which explained the necessity of warping displacement of the cross section in the direction parallel to the axis of twisting, and for the flexure of beams due to transverse loading; the latter allowed understanding of approximations inherent in the simple beam theory of Jakob Bernoulli, Euler, and Coulomb. Heinrich Rudolf Hertz developed solutions for the deformation of elastic solids as they are brought into contact and applied these to model details of impact collisions. Solutions for stress and displacement due to concentrated forces acting at an interior point of a full space were derived by Kelvin and those on the surface of a half space by Boussinesq and Cerruti. In 1863 Kelvin had derived the basic form of the solution of the static elasticity equations for a spherical solid, and this was applied in following years for calculating the deformation of the earth due to rotation and tidal force and measuring the effects of elastic deformability on the motions of the earth's rotation axis.

1.1.3 ASSUMPTIONS OF LINEAR ELASTICITY

In order to evaluate the stresses, strains and displacements in an elasticity problem, one needs to derive a series of basic equations and boundary conditions. During the process of deriving such equations, one can consider all the influential factors, the results obtained will be so complicated and hence practically no solutions can be found. Therefore, some basic assumptions have to be made about the properties of the body considered to arrive at possible solutions. Under such assumptions, we can neglect some of the influential factors of minor importance. The following are the assumptions in classical elasticity.

The Body is Continuous

Here the whole volume of the body is considered to be filled with continuous matter, without any void. Only under this assumption, can the physical quantities in the body, such as stresses, strains and displacements, be continuously distributed and thereby expressed by continuous functions of coordinates in space. However, these assumptions will not lead to significant errors so long as the dimensions of the body are very large in comparison with those of the particles and with the distances between neighbouring particles.

The Body is Perfectly Elastic

The body is considered to wholly obey Hooke's law of elasticity, which shows the linear relations between the stress components and strain components. Under this assumption, the elastic constants will be independent of the magnitudes of stress and strain components.

The Body is Homogenous

In this case, the elastic properties are the same throughout the body. Thus, the elastic constants will be independent of the location in the body. Under this assumption, one can analyse an elementary volume isolated from the body and then apply the results of analysis to the entire body.

The Body is Isotropic

Here, the elastic properties in a body are the same in all directions. Hence, the elastic constants will be independent of the orientation of coordinate axes.

The Displacements and Strains are Small

The displacement components of all points of the body during deformation are very small in comparison with its original dimensions and the strain components and the rotations of all line elements are much smaller than unity. Hence, when formulating the equilibrium equations relevant to the deformed state, the lengths and angles of the body before deformation are used. In addition, when geometrical equations involving strains and displacements are formulated, the squares and products of the small quantities are neglected. Therefore, these two measures are necessary to linearize the algebraic and differential equations in elasticity for their easier solution.

1.1.4 APPLICATIONS OF LINEAR ELASTICITY

The very purpose of application of elasticity is to analyse the stresses and displacements of elements within the elastic range and thereby to check the sufficiency of their strength, stiffness and stability. Although, elasticity, mechanics of materials and structural mechanics are the three branches of solid mechanics, they differ from one another both in objectives and methods of analysis.

Mechanics of materials deals essentially with the stresses and displacements of a structural or machine element in the shape of a bar, straight or curved, which is subjected to tension, compression, shear, bending or torsion. Structural mechanics, on the basis of mechanics of materials, deals with the stresses and displacements of a structure in the form of a bar system, such as a truss or a rigid frame. The structural elements that are not in form of a bar, such as blocks, plates, shells, dams and foundations, they are analysed only using theory of elasticity. Moreover, in order to analyse a bar element thoroughly and very precisely, it is necessary to apply theory of elasticity.

Although bar shaped elements are studied both in mechanics of materials and in theory of elasticity, the methods of analysis used here are not entirely the same. When the element is studied in mechanics of materials, some assumptions are usually made on the strain condition or the stress distribution. These assumptions simplify the mathematical derivation to a certain extent, but many a times inevitably reduce the degree of accuracy of the results obtained. However, in elasticity, the study of bar-shaped element usually does not need those assumptions. Thus the results obtained by the application of elasticity theory are more accurate and may be used to check the appropriate results obtained in mechanics of materials.

While analysing the problems of bending of straight beam under transverse loads by the mechanics of materials, it is usual to assume that a plane section before bending of the beam remains plane even after the bending. This assumption leads to the linear distribution of bending stresses. In the theory of elasticity, however one can solve the problem without this assumption and prove that the stress distribution will be far from linear variation as shown in the next sections.

Further, while analysing for the distribution of stresses in a tension member with a hole, it is assumed in mechanics of materials that the tensile stresses are uniformly distributed across the net section of the member, whereas the exact analysis in the theory of elasticity shows that the stresses are by no means uniform, but are concentrated near the hole; the maximum stress at the edge of the hole is far greater than the average stress across the net section.

The theory of elasticity contains equilibrium equations relating to stresses; kinematic equations relating the strains and displacements; constitutive equations relating the stresses and strains; boundary conditions relating to the physical domain; and uniqueness constraints relating to the applicability of the solution.